Nonlinear oscillations of trapped plasmas

Sh. G. Amiranashvili
General Physics Institute, Moscow 117 942, Russia

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A uniform density ellipsoid is an equilibrium state of a low-energy non-neutral plasma confined in a harmonic trap. Normal modes of this plasma can be detected; they provide a nondestructive diagnostic tool. The low-order quadrupole modes are particularly simple. They can be calculated analytically even in a nonlinear regime. In general, nonlinear coupling leads to the complicated stochastic dynamics. In special cases, regular solutions can be found. It is shown that the behavior of an elongated ellipsoid oriented along the trap symmetry axis is nearly integrable. The study results in simple analytical expressions for frequencies and ellipsoid semiaxes.

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I. INTRODUCTION

A non-neutral plasma [1] is a collection of charged particles identical in sign of the charge. It exhibits many of the properties of neutral plasma, such as the Langmuir waves and the Debye shielding [2]. On the other hand, there are noticeable differences. The non-neutral plasma has excellent confinement properties [3], and can be cooled [4] without recombining positive and negative particles.

We study a confined, low-energy, one-species plasma cloud. This sort of a plasma has a variety of applications, including storage of positrons [5] and antiprotons [6], high-resolution spectroscopic measurements on ions [7], and experiments on vortex dynamics [8] and Coulomb crystals [9,10]. The cloud storage time is enough to observe the thermal equilibrium state and normal modes of oscillations [11]. Being interesting on their own, trapped plasma oscillations provide a nondestructive diagnostic tool. The detection of these modes is used as a probe of plasma properties such as density and temperature [12,13].

Studying the properties of a non-neutral plasma, one is forced to take into account an external trapping field. Let m, v, and q be the particle mass, velocity, and charge, respectively. The external force per particle is

$$\mathbf{F}=-\nabla U_{\text{ext}}+\frac{q}{c}\mathbf{v}\times\mathbf{B}, \tag{1}$$

where $\mathbf{B}=B\hat{\mathbf{z}}$ is the uniform magnetic field, and U_{ext} is the potential of the trapping force. The potential is quadratic, assuming that the plasma size is smaller than the distance to the trap electrodes. If this is the case, the potential is

$$U_{\text{ext}}=\frac{1}{2}m\omega_0^2U_{ij}x_ix_j, \tag{2}$$

where ω_0 is some constant related to the trapping force intensity, and the dimensionless tensor U depends on trap geometry. The summation is over $i,j=1,2$, and 3. In the case of cylindrically symmetric electrodes, the potential energy per particle is

$$U_{\text{ext}}=\frac{1}{2}m\omega_0^2(\alpha x^2+\alpha y^2+z^2),$$

where α is the anisotropy parameter and ω_0 is seen to be the frequency of axial oscillations of a single particle. The Cartesian coordinates x,y, and z are taken with respect to laboratory axes, and measured from the center of the trap.

The two most popular types of traps are the Penning trap [14] and the radio-frequency Paul trap [15]. For the Penning trap the confining force is electrostatic. The parameter $\alpha=-\frac{1}{2}$, and the potential well confines particles in the z direction only. The applied magnetic field provides confinement in a radial direction. For the Paul trap, $B=0$. Charged particles are confined by an inhomogeneous oscillating electric field. For the averaged ponderomotive force, $\alpha=\frac{1}{4}$. Hybrid traps with arbitrary α were also studied [16,17]. As $\alpha\to\infty$, the plasma becomes an infinitely long cylindrical column. The opposite case is a thin disk as $\alpha\to 0$.

Let us assume that the plasma temperature is low, and the density is sufficiently large so that the Debye length is much smaller than the size of the cloud. The plasma therefore has sharp boundaries [18]. The equilibrium state of a cold trapped non-neutral plasma can be described by a cold-fluid theory [19]. The cloud takes the shape of an ellipsoid of revolution. It is uniformly charged and rotates rigidly about the z axis. The theory was verified by the experiment [11,12]. For very low temperatures, the density is not truly constant due to correlations in the particle positions. However, as long as the interparticle spacing is small compared to the cloud dimensions and is small compared to the wavelength of the plasma modes, the cloud can be treated as a constant-density plasma.

The theory can be understood by referring to the known results on the elliptical equilibrium of the rotating massive fluid [20]. The analogy is clear. Both systems have the inverse square law of the interparticle force. The Coriolis force (in the rotating frame of reference) acts like an external magnetic field. In addition, the potential of a centrifugal force is quadratic, as is the potential of the external trapping force. It can be shown that elliptical equilibrium is also possible in a quadrupole trap.

Let β denote the ratio of the axial length to the diameter of the plasma. In this paper, we focus on the behavior of a prolate spheroid with $\beta \gg 1$. Before we proceed, it is appropriate to list the conditions for prolate equilibrium. The aspect ratio β is related [21] to the plasma frequency ω_p and the trap axial frequency ω_0. Assuming $\beta \gg 1$, the relation is

$$\frac{\omega_0^2}{\omega_p^2} = \frac{\ln(2\beta) - 1}{\beta^2}. \tag{3}$$

Therefore, ω_0 should be small as compared to ω_p. This leads to restrictions both on the trap parameters and on the plasma rotation frequency.

Let ω_r denote the rigid rotation frequency. In the case of the Penning trap, the expression for ω_r is [22]

$$\omega_p^2 = -2\omega_r(\omega_c + \omega_r), \tag{4}$$

where $\omega_c = qB/mc$ is the particle cyclotron frequency. The plasma frequency attains its maximum value at $\omega_r = -\omega_c/2$. This limit is often called the Brillouin flow. Then ω_p and ω_c are of the same order, and the inequality $\beta \gg 1$ requires $\omega_0 \ll \omega_c$. The opposite case of a rare plasma with relatively small ω_p corresponds to the guiding-center limit. Then ω_r is of the order of ω_p^2/ω_c, and is small compared with ω_c. The condition for the prolate equilibrium is $\omega_0 \ll \omega_p \ll \omega_c$.

In the case of the gibrid Paul trap with $\omega_c = 0$, the equation for the rigid rotation frequency is

$$\omega_p^2 = (2\alpha + 1)\omega_0^2 - 2\omega_r^2, \tag{5}$$

and the inequality $\alpha \gg 1$ is required. Note that in the absence of a magnetic field, the thermodynamic equilibrium corresponds to a static ellipsoid. We also consider a more general hydrodynamic equilibrium with $\omega_r \neq 0$. The rigid-rotor frequency must be less than $\alpha^{1/2}\omega_0$ (trap radial frequency). In the special case of fast rotation, both $\alpha^{1/2}\omega_0$ and ω_r are large as compared to ω_p. Then we assume that ω_p is still high as compared to ω_0.

Let us now change from equilibrium to dynamics. Normal modes of an elliptical plasma allow an analytical treatment [23]. Each mode is described by two positive integers (l and m), with $l \geq m$. The index m refers to a familiar $\exp(im\theta)$ azimuthal dependence of perturbed quantities. The index l describes the variation of the plasma boundary along the z direction. The $l=1$ modes correspond to the center of mass motion. The quadrupole $l=2$ modes are of particular interest, because they provide information on the plasma density and aspect ratio. In this paper, we consider the nonlinear behavior of quadrupole oscillations.

A goal of this research is to explore a fairly simple approach to quadrupole modes first suggested by Dubin [24]. Following him, we derive a set of nine equations describing the ellipsoid orientation, the semiaxes, and the internal plasma motion. Then we calculate two different types of $l=2$ plasma oscillations, namely, (2,0) and (2,2) modes. The former one corresponds to cylindrically symmetric oscillations in the length and radius of the ellipsoid. The latter one corresponds to a plasma that has been flattened to a triaxial ellipsoid with a principal axis oriented along $\hat{\mathbf{z}}$.

We find that these quadrupole modes are equivalent to a set of three coupled nonlinear oscillators. Then we employ an inequality $\beta \gg 1$. The oscillator corresponding to the asymmetric mode turns out to be separated. What is more, the solution has a simple exact expression in terms of elementary functions. We are, therefore, left with a full description of the finite amplitude version of asymmetric mode. Then we turn to coupled radial and axial oscillations. Fortunately, radial oscillations of the prolate ellipsoid are at a much higher frequency than the axial ones. The resonance does not occur, and the system is nearly integrable. To solve the equations, we use the method of averaging in its canonical (Hamiltonian) form. Finally, we are left with simple analytical expressions for nonlinear modes and their frequencies.

II. BASIC EQUATIONS

In this section we consider the general set of hydrodynamic equations for plasma motion. In the case of quadrupole oscillations, this is reduced to a relatively simple set of ordinary differential equations. The reduction procedure is the generalization of the elegant technique first developed to describe the behavior of a self-gravitating fluid mass.

Let n and $\mathbf{v}$ be the number density and velocity of the plasma. They are related by the continuity equation

$$\frac{\partial n}{\partial t} + \nabla(n\mathbf{v}) = 0. \tag{6}$$

The motion of a cold plasma is expressed by the ideal pressureless Euler equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = \frac{q}{m}\mathbf{E} + \frac{1}{m}\mathbf{F}, \tag{7}$$

where the applied trapping force $\mathbf{F}$ is given by Eqs. (1) and (2). The space-charge electric field $\mathbf{E} = -\nabla\phi$ is related to the number density by the Poisson equation

$$\triangle\phi = -4\pi q n. \tag{8}$$

As for a plasma boundary, the condition for a smooth surface $S(\mathbf{x},t) = \text{const}$ to be the boundary of the moving fluid is

$$\frac{\partial S}{\partial t} + (\mathbf{v}\nabla)S = 0. \tag{9}$$

Equations (6)–(9) provide a full description of the plasma motion. They are related to the laboratory Cartesian frame (x,y,z). In some instances it is also useful to consider a body frame reference (x',y',z'). The coordinates of the same point in these frames are related by $x_i' = R_{ij}x_j$, where R is some orthogonal matrix. This can be expressed explicitly in terms of the Euler angles. Now we seek a special solution of Eqs. (6)–(9), which is subject to following conditions.

(i) A plasma takes the form of an ellipsoid. With respect to laboratory axes, the plasma boundary is given by

$$G_{ij}x_ix_j = 1,$$

where the symmetric, positive-definite tensor G describes the shape and orientation of the ellipsoid. With respect to the body axes (primed variables), the plasma surface satisfies the equation

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}=1,$$

where $a(t)$, $b(t)$, and $c(t)$ are unknown functions. Then $G'=\mathrm{diag}(a^{-2},b^{-2},c^{-2})$, and, in accordance with the standard transformation rule $G=R^TG'R$, where R^T denotes a matrix transposed to R.

(ii) The number density is uniform, and depends on time only: $n=n(t)$. For the ellipsoidal plasma, it can be put up in the form

$$n(t)=\frac{3N}{4\pi abc}, \tag{10}$$

where N is particle number. What is more, it is well known that a space-charge electric field within a uniformly charged ellipsoid is the linear function of position. It can be expressed in the form

$$E_i=4\pi qn\Lambda_{ij}x_j, \tag{11}$$

where the dimensionless tensor Λ depends on G. The explicit expression for the electric field can be derived initially along the body axes. In the body frame the space-charge potential within a uniformly charged ellipsoid is

$$\phi=\pi qnabc\int_0^\infty\left\{1-\frac{x'^2}{a^2+\xi}-\frac{y'^2}{b^2+\xi}-\frac{z'^2}{c^2+\xi}\right\}\frac{d\xi}{\sqrt{(a^2+\xi)(b^2+\xi)(c^2+\xi)}}. \tag{12}$$

Following Ref. [20], we introduce quantities I and A_i, so to put Eq. (12) into the form

$$\phi=\pi qn(I-A_1x'^2-A_2y'^2-A_3z'^2). \tag{13}$$

In line with Eq. (11), we see that in the body frame $\Lambda'=\frac{1}{2}\mathrm{diag}(A_1,A_2,A_3)$. In the lab frame $\Lambda=R^T\Lambda'R$. The functions $A_i(a,b,c)$ can be written in terms of elliptical integrals, as listed in the Appendix.

(iii) The fluid velocity is a linear function of position,

$$v_i=W_{ij}x_j, \tag{14}$$

where $W(t)$ is 3×3 matrix (velocity tensor). It can be easily calculated in the body frame. To begin with, let us assume that in the body frame the plasma rotates with some given frequency vector $\mathbf{\Omega}'$. Therefore $\mathbf{v}'=\mathbf{\Omega}'\times\mathbf{x}'$. In other words $W'=\hat{\Omega}'$, where $\hat{\Omega}'$ is an antisymmetric pseudotensor of rank 2, which is dual of $\mathbf{\Omega}'$. In Cartesian coordinates [25],

$$\hat{\Omega}=\begin{pmatrix}0 & -\Omega'_z & \Omega'_y\\ \Omega'_z & 0 & -\Omega'_x\\ -\Omega'_y & \Omega'_x & 0\end{pmatrix}. \tag{15}$$

As to a general flow with linear velocity within the ellipsoid, the expression for W' can be written as a simple generalization of the case of pure rotation [26],

$$W'=Q\hat{\Omega}'Q^{-1}+\frac{dQ}{dt}Q^{-1}, \tag{16}$$

where $Q=\mathrm{diag}(a,b,c)$.

We see that the internal plasma motion depends on semiaxes, and is characterized by the vector $\mathbf{\Omega}'$. In the case of constants a, b, and c, the particles follow closed orbits. The rotation period is $2\pi/\Omega'$. For time variant semiaxes, the particles do not return to their original position, and $\mathbf{\Omega}'$ is not the frequency vector. In any event, $\mathbf{\Omega}'$ can be expressed in terms of vorticity.

To obtain the expression for W one should employ the corresponding transformation rule. Note that the behavior of the velocity matrix is more complicated, as compared to the other tensors. The point is that the rotation from the lab frame to the body frame depends on time. The dependence cannot be ignored when calculating particles velocity. Let $\mathbf{x}$ and $\mathbf{x}'$ be the coordinates of the same particle. Taking the time derivative of $\mathbf{x}'=R\mathbf{x}$, one can derive the transformation rule

$$W=R^T(W'+\hat{\omega}')R, \tag{17}$$

where $\hat{\omega}'=R(dR^T/dt)$ is an antisymmetric tensor. Therefore it is dual of some vector $\boldsymbol{\omega}'$, which is recognized as the rotation frequency vector of the body frame along the body axes. The desired expression for W is obtained by inserting Eq. (16) into Eq. (17).

It appears that the listed conditions do not contradict each other. What is more, the basic equations reduce to a self-consistent system of ordinary differential equations with no spatial dependence. To show this, one should substitute n, $\mathbf{v}=W\mathbf{x}$ and $S=\mathbf{x}^TG\mathbf{x}$, into Eqs. (6)–(9), and eliminate the $\mathbf{x}$ dependence. The calculations are left to the reader. Details are similar to those of Ref. [24]. Equation (7) results in a single matrix equation for $Q=\mathrm{diag}(a,b,c)$. The remaining equations are satisfied identically. After some effort we are left with

$$\frac{d^2Q}{dt^2}+2\left(\hat{\omega}'\frac{dQ}{dt}+\frac{dQ}{dt}\hat{\Omega}'\right)+\left(\frac{d\hat{\omega}'}{dt}+\hat{\omega}'^2\right)Q+Q\times\left(\frac{d\hat{\Omega}'}{dt}+\hat{\Omega}'^2\right)+2\hat{\omega}'Q\hat{\Omega}'+\omega_c\hat{b}'\times\left(\hat{\omega}'Q+\frac{dQ}{dt}+Q\hat{\Omega}'\right)+\omega_0^2U'Q=\frac{4\pi q^2n}{m}\Lambda'Q, \tag{18}$$

where $U'=RUR^T$, $\hat{b}'=R\hat{b}R^T$, and the antisymmetric tensor $\hat{b}$ is dual of a unit vector parallel to the magnetic field.

Therefore, we have nine equations for the ellipsoid semiaxes, $\boldsymbol{\omega}'$ and $\mathbf{\Omega}'$. In the case of trapped plasma, these were first investigated by Dubin [24]. Equation (18) closely resembles the equation of motion for a self-gravitating ellipsoidal fluid mass originally derived by Riemann (see [20] and references cited therein). However, the presence of the external trapping field significantly complicates the problem. The point is that $\hat{b}'$ as well as U' depends on R components.

Therefore, one is forced to consider an explicit expression for R. It can be taken in terms of the Euler angles [27], and Eq. (18) must be supplemented by three Euler equations for $\boldsymbol{\omega}'$.

The difficulties can be overcome when considering equilibrium configurations with $d/dt=0$. As to the nonlinear dynamics, there seems to be only one case which promises significant progress. Assume that the ellipsoid is oriented with a principal axis along the magnetic field, which is parallel to the trap symmetry axis. Now the rotation from the lab frame to the body frame does not change either $\hat{b}$ or U. This eliminates the Euler angles. The vectors $\boldsymbol{\omega}'$ and $\boldsymbol{\Omega}'$ should also be oriented along the magnetic field (the z axis). Under these assumptions the system is reduced to a five relatively simple equations. The diagonal elements of Eq. (18) are

$$\frac{d^2a}{dt^2}+(\alpha\omega_0^2-\omega_z'^{\,2}-\Omega_z'^{\,2})a-2\omega_z'\Omega_z'b-\omega_c \times(\omega_z'a+\Omega_z'b)=\frac{\sigma}{2bc}A_1, \tag{19}$$

$$\frac{d^2b}{dt^2}+(\alpha\omega_0^2-\omega_z'^{\,2}-\Omega_z'^{\,2})b-2\omega_z'\Omega_z'a-\omega_c \times(\Omega_z'a+\omega_z'b)=\frac{\sigma}{2ac}A_2, \tag{20}$$

$$\frac{d^2c}{dt^2}+\omega_0^2c=\frac{\sigma}{2ab}A_3, \tag{21}$$

where $\sigma=3q^2N/m$.

Four out of six off-diagonal elements are identically zero. By taking the sum and difference of the remaining two, we bring them to the integrable forms

$$\frac{d}{dt}\left[\left(\frac{\omega_c}{2}+\omega_z'+\Omega_z'\right)(a+b)^2\right]=0,$$

$$\frac{d}{dt}\left[\left(\frac{\omega_c}{2}+\omega_z'-\Omega_z'\right)(a-b)^2\right]=0. \tag{22}$$

This allows us to eliminate ω_z' and Ω_z'. Finally we are given the three second-order differential equations for the ellipsoid semiaxes. In general the system is not integrable. The only known integrable case corresponds to the ellipsoid of revolution in the guiding center limit [24]. In the remaining part of the article, we describe another integrable case corresponding to a triaxial prolate ellipsoid with $a,b\ll c$.

Before we proceed, it is appropriate to derive equilibrium conditions (3)–(5). For the rigid-rotor equilibrium, $\Omega_z'=0$ and $\omega_z'=\omega_r$. By dropping the time derivatives and taking the sum of Eqs. (19)–(21), we obtain

$$\omega_p^2=-2\omega_r(\omega_c+\omega_r)+(2\alpha+1)\omega_0^2.$$

Now we recall that for the Penning trap $\alpha=-\frac{1}{2}$ and for the Paul trap $\omega_c=0$.

As to the plasma aspect ratio, in the case of the ellipsoid of revolution, A_i can be expressed in terms of elementary functions. By doing this and assuming that $a=b\ll c$, one can easily derive Eq. (3) from Eq. (21).

III. NONLINEAR OSCILLATIONS

The goal of this section is to describe the behavior of the prolate ellipsoid with the longest principal axis oriented along the magnetic field. To begin with, we note that Eqs. (22) suggest the introduction of a notation

$$\omega_\pm=\frac{\omega_c}{2}+\omega_z'\pm\Omega_z'.$$

Then, by taking the sum and the difference of Eqs. (19) and (20), we put them in the forms

$$\frac{d^2}{dt^2}(a+b)+(\tilde{\omega}_0^2-\omega_+^2)(a+b)=\frac{\sigma}{2abc}(aA_1+bA_2), \tag{23}$$

$$\frac{d^2}{dt^2}(a-b)+(\tilde{\omega}_0^2-\omega_-^2)(a-b)=\frac{\sigma}{2abc}(aA_1-bA_2), \tag{24}$$

where

$$\tilde{\omega}_0^2=\alpha\omega_0^2+\frac{1}{4}\omega_c^2$$

depends on the trap parameters only.

The quantities A_i can be expressed in terms of elliptical integrals. In the limit that one principal axis becomes very long, these expression are simplified. By taking a/b finite and $c\to\infty$ in line with Eq. (12), we find

$$A_1=\frac{2b}{a+b},\quad A_2=\frac{2a}{a+b},\quad A_3=0. \tag{25}$$

Inserting $A_{1,2}$ into Eqs. (23) and (24), and introducing new variables

$$u=\frac{a+b}{\sqrt{2}},\quad v=\frac{a-b}{\sqrt{2}},$$

the following two equations are found:

$$\frac{d^2u}{dt^2}+(\tilde{\omega}_0^2-\omega_+^2)u=\frac{\sigma}{cu} \tag{26}$$

$$\frac{d^2v}{dt^2}+(\tilde{\omega}_0^2-\omega_-^2)v=0. \tag{27}$$

According to Eq. (22) the quantities $\omega_\pm$ can be expressed as

$$\omega_+=\tilde{\omega}_0\frac{\mu}{u^2},\quad \omega_-=\tilde{\omega}_0\frac{\nu}{v^2}, \tag{28}$$

where μ and ν are the constants of integration.

System (26)–(28) must be accomplished by the equation for $c(t)$. Unfortunately, the approximation $A_3=0$ does not lead to a reasonable conclusion. We need a more accurate expression, containing a/c and b/c. Surprisingly, the calculation of such an expression involves rather sophisticated mathematical techniques. The details can be found in the Appendix. The resulting equation is

$$\frac{d^2c}{dt}+\omega_0^2c=\frac{\sigma}{c^2}\left(\ln\frac{2\sqrt{2}c}{u}-1\right). \tag{29}$$

Now we are able to give a full description of nonlinear oscillations. Recall that the (2,2) mode corresponds to a spheroid that has been flattened to a triaxial ellipsoid, with a principal axis oriented along $\hat{\mathbf{z}}$. This mode corresponds to Eq. (27), and is separated from the others even in a nonlinear regime. The equation can be integrated in terms of elementary functions. The period of oscillations is independent of their amplitude. For $\nu=0$, the oscillations are harmonic. Otherwise, $v(t)$ does not change its sign. One can put

$$v_{\min}^2\leqslant v^2(t)\leqslant v_{\max}^2,\quad v_{\max}^2v_{\min}^2=\nu^2.$$

Taking $dv/dt(0)=0$, $v(0)=v_{\max}$, the following solution is found:

$$v^2(t)=\frac{v_{\max}^2+v_{\min}^2}{2}+\frac{v_{\max}^2-v_{\min}^2}{2}\cos 2\,\tilde{\omega}_0t. \tag{30}$$

For $\nu=0$ the quantity $v_{\min}=0$; then positive and negative solutions result in $v=v_{\max}\cos\tilde{\omega}_0t$. It is of interest that even in the limit of small oscillations there are two different frequencies: $\omega=2\tilde{\omega}_0$ for $\nu\neq 0$, and $\omega=\tilde{\omega}_0$ for $\nu=0$. The latter case corresponds to a spheroid.

Instead of Eq. (27), one may prefer to consider a Hamiltonian

$$H=\frac{\dot{v}^2}{2}+\frac{\tilde{\omega}_0^2}{2}\left(v^2+\frac{\nu^2}{v^2}\right).$$

Since the frequency of the mode is unaffected by the amplitude, it is immediately obvious that $J_v=H/(2\tilde{\omega}_0)$ is an adiabatic invariant. It can be put into the form

$$J_v=\frac{\tilde{\omega}_0}{4}(v_{\max}^2+v_{\min}^2).$$

The quantity is conserved even if the trap parameters are time variant.

Let us turn to the (2,0) mode corresponding to cylindrically symmetric oscillations in length and in radius of the ellipsoid. It is described by Eqs. (26) and (29).

In general, a nonlinear coupling between the oscillations in length and in radius of the ellipsoid may result in a complicated stochastic dynamics. Fortunately, the case of elongated spheroid is much more simple. The conditions of prolate equilibrium, which were listed in Sec. I, show that ω_0 is small as compared with $\tilde{\omega}_0$. Therefore the radial oscillations are at a much higher frequency than the axial oscillations. In other words, one can consider $c(t)$ as a constant parameter when calculating $u(t)$. This makes Eq. (26) integrable. Taking $u(0)=u_{\max}$ and $du/dt(0)=0$, the following solution is found:

$$\tilde{\omega}_0t=\int_{u_{\max}}^{u}\frac{u\,du}{\sqrt{(u_{\max}^2-u^2)\left(u^2-\frac{\mu^2}{u_{\max}^2}\right)+\frac{2\sigma}{c\tilde{\omega}_0^2}u^2\ln\frac{u}{u_{\max}}}}. \tag{31}$$

Therefore we are left with a direct integral representation for $u(t)$. The frequency of the small oscillations reads $\omega^2=4\tilde{\omega}_0^2-\omega_p^2$. For $\mu=0$ we have $\omega_p^2=2\tilde{\omega}_0^2$ and $\omega=\omega_p$. This case corresponds to a spheroid at the Brillouin limit [28].

Farther simplification can be achieved in the guiding center limit. We will show this by taking the advantage of the Hamiltonian technique. Equation (26) is related to the Hamiltonian

$$H=H_0+H_{\text{int}},$$

where

$$H_0=\frac{\dot{u}^2}{2}+\frac{\tilde{\omega}_0^2}{2}\left(u^2+\frac{\mu^2}{u^2}\right),\quad H_{\text{int}}=\frac{\sigma}{c}\ln\frac{2\sqrt{2}c}{u}.$$

The ratio H_{int}/H_0 is of order $\omega_p^2/\tilde{\omega}_0^2$. For the Penning trap, the ratio is small in the guiding center limit. For the Paul trap, it is small for the fast rotating spheroid. If this is the case, the right-hand side of Eq. (26) is a small perturbation. By dropping it, one can write the expression for $u(t)$ much as for $v(t)$. The only difference is that negative solutions are not allowed. For example, for small μ the solution takes the form $u=u_{\max}|\cos\tilde{\omega}_0t|$.

Now our goal is to take into account the corrections concerned with the right-hand side of Eq. (26). We will see that a nonlinear coupling of radial and axial oscillations results in some frequency shift.

Let us introduce new canonical variables (J,ψ) instead of $(\dot{u},u)$. The transformation is described by the following generating function:

$$S(J,u)=\tilde{\omega}_0\int\sqrt{\frac{4J}{\tilde{\omega}_0}-u^2-\frac{\mu^2}{u^2}}du.$$

The equations $\psi=\partial S/\partial J$ and $\dot{u}=\partial S/\partial u$ yield

$$u^2=\frac{2J}{\tilde{\omega}_0}+\cos\psi\sqrt{\left(\frac{2J}{\tilde{\omega}_0}\right)^2-\mu^2}, \tag{32}$$

$$2\tilde{\omega}_0J=\frac{\dot{u}^2}{2}+\frac{\tilde{\omega}_0^2}{2}\left(u^2+\frac{\mu^2}{u^2}\right). \tag{33}$$

This puts the Hamiltonian into the form

$$H=2\tilde{\omega}_0J+\frac{\sigma}{c}\ln\frac{2\sqrt{2}c}{u(J,\psi)},$$

where $u(J,\psi)$ should be taken from Eq. (32). Since $H_{\rm int}$ describes a small perturbation of the periodic motion, one can average $H_{\rm int}$ over ψ:

$$H_{\rm int}\mapsto\langle H_{\rm int}\rangle=\frac{\sigma}{c}\ln\frac{2\sqrt{2}c}{\sqrt{(1/\tilde{\omega}_0)J+(1/2)|\mu|}}. \tag{34}$$

The procedure is the canonical version of the method of averaging [29]. It puts the radial oscillations into an integrable form

$$\frac{dJ}{dt}=0,\quad \frac{d\psi}{dt}=2\tilde{\omega}_0-\frac{\sigma}{c(2J+|\mu|\tilde{\omega}_0)}.$$

The quantity J is an adiabatic invariant. By taking $u_{\rm min}\leqslant u(t)\leqslant u_{\rm max}$ in line with expression (33) for $J(\dot{u},u)$, the following equations are found:

$$u_{\rm max}^2u_{\rm min}^2=\mu^2,\quad J=\frac{\tilde{\omega}_0}{4}(u_{\rm max}^2+u_{\rm min}^2).$$

Employing Eq. (32) for $u(J,\psi)$ and assuming $du/dt(0)=0$, $u(0)=u_{\rm max}$, we derive the final expression for $u(t)$,

$$u^2(t)=\frac{u_{\rm max}^2+u_{\rm min}^2}{2}+\frac{u_{\rm max}^2-u_{\rm min}^2}{2}\cos\omega t, \tag{35}$$

where

$$\omega=2\tilde{\omega}_0-\frac{2\sigma}{c(u_{\rm max}+u_{\rm min})^2\tilde{\omega}_0}. \tag{36}$$

The method of averaging is valid as far as the frequency shift is small compared with $2\tilde{\omega}_0$. Then Eq. (36) can be put into the equivalent form

$$\omega^2=4\tilde{\omega}_0^2-\frac{8\sigma}{c(u_{\rm max}+u_{\rm min})^2}. \tag{37}$$

Equation (37) is the best choice. Note, that for the small oscillations $u_{\rm max}+u_{\rm min}$ can be replaced by twice the equilibrium value of u. Then Eq. (37) takes the form $\omega^2=4\tilde{\omega}_0^2-\omega_p^2$, which is valid even in the case of comparably valued ω_p and $\tilde{\omega}_0$. It follows that Eq. (37) is a reasonable approximation near the Brillouin limit. Moreover, it is valid for a strongly nonlinear regime with $u_{\rm max}\gg u_{\rm min}$.

In reality, one may prefer to average the radial frequency over the oscillations in length of the spheroid. Then Eq. (37) should be averaged as well. The simplest approximation is that $c(t)$ varies harmoniously from minimal to maximal values. Then one can see that $\langle 1/c\rangle$ should be replaced by $(c_{\rm max}c_{\rm min})^{-1/2}$. Therefore, Eq. (37) takes the form

$$\omega^2=4\tilde{\omega}_0^2-\frac{8\sigma}{(c_{\rm max}c_{\rm min})^{1/2}(u_{\rm max}+u_{\rm min})^2}. \tag{38}$$

This simple relation is of special interest, because it directly links the radial frequency with the ellipsoid shape and particle number.

Now let us turn to more detailed investigation of Eq. (29) for the length of the spheroid. The related Hamiltonian is

$$H=H_c+H_{\rm int}=\left(\frac{\dot{c}^2}{2}+\frac{\omega_0^2c^2}{2}\right)+\frac{\sigma}{c}\ln\frac{2\sqrt{2}c}{u}.$$

The quantities H_c and $H_{\rm int}$ are of the same order. Nevertheless, the latter term oscillates rapidly and should be averaged. Note that Eq. (29) contains $u(t)$ under a logarithmic sign. Therefore, it is not necessary to apply sophisticated equation (31) in order to average $H_{\rm int}$ over the radial oscillations. One can use the guiding center expression (34) instead. The result can be put into the form

$$\langle H_{\rm int}\rangle=\frac{\sigma}{c}\ln\frac{4\sqrt{2}c}{u_{\rm max}+u_{\rm min}}.$$

Therefore, we are given the following equation:

$$\frac{d^2c}{dt^2}+\omega_0^2c=\frac{\sigma}{c^2}\left(\ln\frac{4\sqrt{2}c}{u_{\rm max}+u_{\rm min}}-1\right). \tag{39}$$

The dynamics of $c(t)$ is integrable, given by the integral

$$\omega_0t=\int_{c_{\rm max}}^{c}\frac{dc}{\sqrt{2[U_{\rm eff}(c_{\rm max})-U_{\rm eff}(c_{\rm max})]}},$$

with

$$U_{\rm eff}(c)=\frac{c^2}{2}+\frac{\sigma}{\omega_0^2c}\ln\frac{4\sqrt{2}c}{u_{\rm min}+u_{\rm max}}.$$

The equilibrium value $c=c_0$ is obtained from Eq. (39) by dropping the time derivatives. This leads to a simple generalization of Eq. (3). The frequency of small oscillations is $\omega^2=3\omega_0^2-(1/c_0^3)\sigma$. The latter term is less then the former by a factor of $\ln\beta$.

To supplement the theory, we obtained several numerical solutions of the exact Eqs. (21), (23), and (24). In the remaining part of this section the expressions are written in terms of the dimensionless variables. Times are normalized by $1/\tilde{\omega}_0$, and distances by $(\sigma/\tilde{\omega}_0^2)^{1/3}$. The normalized variables are denoted by an overbar.

We start with the spheroidal plasma. In the case or radial oscillations, Eq. (24) is satisfied identically. In addition, the quantities A_i can be expressed in terms of elementary functions. By doing this one can put Eqs. (21) and (23) into the forms

$$\frac{d\overline{u}^2}{d\overline{t}^2}+\overline{u}-\frac{\overline{\mu}^2}{\overline{u}^3}=\frac{1}{\overline{c}\overline{u}}\left(\frac{1}{e^2}+\frac{1-e^2}{2e^3}\ln\frac{1+e}{1-e}\right), \tag{40}$$

$$\frac{d\overline{c}^2}{d\overline{t}^2}+\epsilon\overline{c}=\frac{1}{\overline{c}^2}\left(\frac{1}{2e^3}\ln\frac{1+e}{1-e}-\frac{1}{e^2}\right), \tag{41}$$

where $\epsilon=\omega_0^2/\tilde{\omega}_0^2$, and the eccentricity $e=\sqrt{1-\overline{u}^2/(2\overline{c}^2)}$.

Equations (40) and (41) have been integrated numerically for several cases. Once the equilibrium values of $\overline{u(t)}$ and

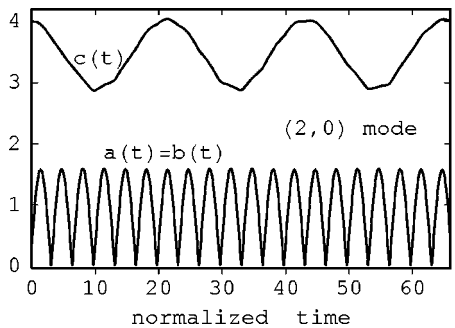

FIG. 1. A numerical solution of Eqs. (40) and (41) for oscillations in radius and length of a prolate plasma spheroid. The upper curve is one-half the normalized plasma length $\bar{c}(\bar{t})$, and the lower curve is the radius $\bar{a}(\bar{t})=\bar{b}(\bar{t})$. An equilibrium spheroid with $\bar{u}_0=0.5$ and $\bar{c}_0=4$ was perturbed by imposing $d\bar{u}/d\bar{t}(0)=2$. The possibility of a two-time-scale approach is clearly observable. The oscillations are highly nonlinear, but the system shows a coherent (integrable) behavior.

$\bar{c}(\bar{t})$ were specified, we calculated corresponding $\bar{\mu}$ and ϵ. Then the radial oscillations of the spheroid were excited by setting $d\bar{u}/d\bar{t}(0)$ nonzero.

A typical example of a numerical solution is shown in Fig. 1. In this case the values of the semiaxes were chosen in such a way that ω_p and $\tilde{\omega}_0$ are of the same order. Therefore, the equilibrium spheroid is close to the Brillouin limit. Nevertheless the large difference in the frequencies of the radial and axial oscillations is clearly observable. A major initial value of $d\bar{u}/d\bar{t}(0)$ leads to a highly nonlinear motion with $u_{\max}/u_{\min}\approx 10^2$. Nevertheless, the nonlinear coupling is small. This suggests that the system is nearly integrable.

For each numerical solution we determined the maximal and minimal values of $\bar{u}(\bar{t})$ and $\bar{c}(\bar{t})$. Then the averaged frequency of the radial oscillations was compared with that predicted by a dimensionless version of Eq. (38). Some results are shown in Fig. 2. For clarity we present the predicted values of the frequency as smooth curves. The reader should keep in mind that in reality all data were discrete sets of points.

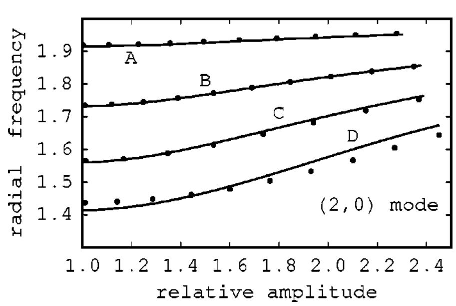

FIG. 2. A normalized frequency of the radial oscillations of a prolate plasma spheroid (dots) is compared with that (solid lines) predicted by Eq. (38). The relative amplitude is $\bar{u}_{\max}/\bar{u}_0$. Equilibrium spheroids with (A) $\bar{u}_0=1$ and $\bar{c}_0=6$, (B) $\bar{u}_0=0.4$ and $\bar{c}_0=8$, (C) $\bar{u}_0=0.5$ and $\bar{c}_0=8$, and (D) $\bar{u}_0=0.5$ and $\bar{c}_0=4$ were perturbed by setting $d\bar{u}/d\bar{t}(0)\neq 0$. Case (D) is close to the Brillouin limit. Nevertheless, a simple guiding center formula provides a reasonable agreement with the computing.

The asymmetric mode of the spheroid was exited by setting nonzero $d\bar{v}/d\bar{t}(0)$. Now the quantities A_i are calculated in a more cumbersome way, and the computing involves Eq. (24) in line with Eqs. (21) and (23). The theoretical value of the frequency $\bar{\omega}=1$ was found to be in a good agreement with that obtained from the computing.

Several examples of triaxial equilibrium ellipsoids were considered in addition to the spheroidal plasma. In summary, Eq. (38) was found to be a good approximation for the radial oscillations, either in the guiding center regime or in a regime which is close to the Brillouin limit.

IV. CONCLUSION

We have studied oscillations of the one component low-energy plasma confined in a harmonic Penning trap or in a radio-frequency Paul trap. The plasma equilibrium state is a uniform density ellipsoid of revolution. Normal modes corresponding to quadrupole perturbations of this plasma are of particular interest, because they provide a nondestructive diagnostic tool.

The quadrupole oscillations of a trapped plasma can be calculated analytically even in the nonlinear regime. The theory is based on the analogy between the one-component plasma and the massive fluid. The oscillations are described by a relatively simple Hamiltonian system, originally derived by Dubin [24]. In general the dynamics is stochastic. Nevertheless, in some particular cases the system can be nearly integrable. We have found and studied in detail a case of regular behavior.

It is of interest that a reduced version of the Hamiltonian corresponding to an infinite elliptical column turns out to be integrable in quadratures [30]. This suggests considering the dynamics of the elongated ellipsoid.

We investigated the possibility for a semiaxis to be large as compared to the others axes. It appears that prolate equilibrium can be observed in the Penning trap both in the Brillouin zone and in the guiding-center limit. For the Paul trap, the elongated spheroid exists only if the radio-frequency trapping force is supplemented by the electrostatic one, so as to provide strong anisotropy.

Then we have considered the dynamics of an ellipsoid with a principal axis oriented along the magnetic field. The system is equivalent to the three coupled nonlinear oscillators. One oscillator corresponds to the asymmetric radial perturbation of the spheroid. The others describe symmetric radial and axial oscillations.

In the limit of an elongated ellipsoid, the matter is considerably simplified. The asymmetric mode becomes separated from the others, and is reduced to quadratures. The other two oscillators are still coupled, but the radial oscillations have much higher frequencies than the axial ones. We solved this problem by means of the method of averaging. The study results in simple analytical expressions for the frequencies and the ellipsoid semiaxes.

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APPENDIX

In this section we calculate A_i in the limit that one principal axis becomes very long. A first approximation is given by Eq. (25). It corresponds to an infinite column with an elliptical cross section. This approximation is sufficient for radial oscillations, but is inadequate to give a description of axial oscillations. Here the calculation is carried out to a higher accuracy.

Recall that in terms of A_i, Eq. (12) for the space-charge potential within a uniformly charged ellipsoid takes the form of Eq. (13), where

$$I=abc\int_0^\infty \frac{d\xi}{\sqrt{(a^2+\xi)(b^2+\xi)(c^2+\xi)}}.$$

Only $I(a,b,c)$ needs further consideration, because of the relations [20]

$$A_1=\frac{I}{a^2}-\frac{1}{a}\frac{\partial I}{\partial a},\quad A_2=\frac{I}{b^2}-\frac{1}{b}\frac{\partial I}{\partial b},\quad A_3=\frac{I}{c^2}-\frac{1}{c}\frac{\partial I}{\partial c}.$$

One can assume that $a<b<c$ and define $\delta\in[0,\pi/2]$ and $k\in[0,1]$, so that

$$\frac{a}{c}=\sin\delta,\quad \frac{c^2-b^2}{c^2-a^2}=k^2.$$

By replacing the variables

$$\xi\mapsto\theta,\quad 1+\frac{\xi}{c^2}=\frac{\cos^2\delta}{\sin^2\theta},$$

we put the expression for I into the form

$$I=\frac{2ab}{\cos\delta}F(\pi/2-\delta,k),$$

where

$$F(\theta,k)=\int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

is an elliptic integral. By putting $c\to\infty$, we see that both quantities δ and $k'=\sqrt{1-k^2}$ are small. The further calculation is not immediate. The point is that the regular expansion for $F(\theta,k)$ near the point $(\pi/2,1)$ can be obtained only if one of the two arguments is fixed.

To proceed, we employ a relation [31]

$$F(\theta,k)+F(\theta^*,k)=F(\pi/2,k),$$

which is valid if

$$k'\tan\theta\tan\theta^*=1.$$

Fortunately, in our case $\tan\theta^*=a/\sqrt{b^2-a^2}$ is independent of c. Then,

$$I=\frac{2ab}{\cos\delta}\left[F(\pi/2,k)-F\left(\arcsin\frac{a}{b},k\right)\right].$$

Now both integrals can be expanded in a standard manner. The calculation results in

$$\frac{I}{2ab}=\ln\frac{4c}{a+b}+\frac{a^2+b^2}{4c^2}\left(\ln\frac{4c}{a+b}-1\right)+\frac{ab}{4c^2}+\cdots.$$

Then

$$A_1=\frac{2b}{a+b}\left(1+\frac{a^2+b^2}{4c^2}\right)-\frac{b^2}{2c^2}-\frac{ab}{c^2}\left(\ln\frac{4c}{a+b}-1\right),$$

$$A_2=\frac{2a}{a+b}\left(1+\frac{a^2+b^2}{4c^2}\right)-\frac{a^2}{2c^2}-\frac{ab}{c^2}\left(\ln\frac{4c}{a+b}-1\right),$$

$$A_3=\frac{2ab}{c^2}\left(\ln\frac{4c}{a+b}-1\right).$$

These are the desired generalizations of Eqs. (25). The expression for A_3 is essential to the calculation of axial oscillations.

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